Transformation $\vec{r}, \vec{V}$ to Orbital elements $a, e, i, \Omega, \omega, \nu$

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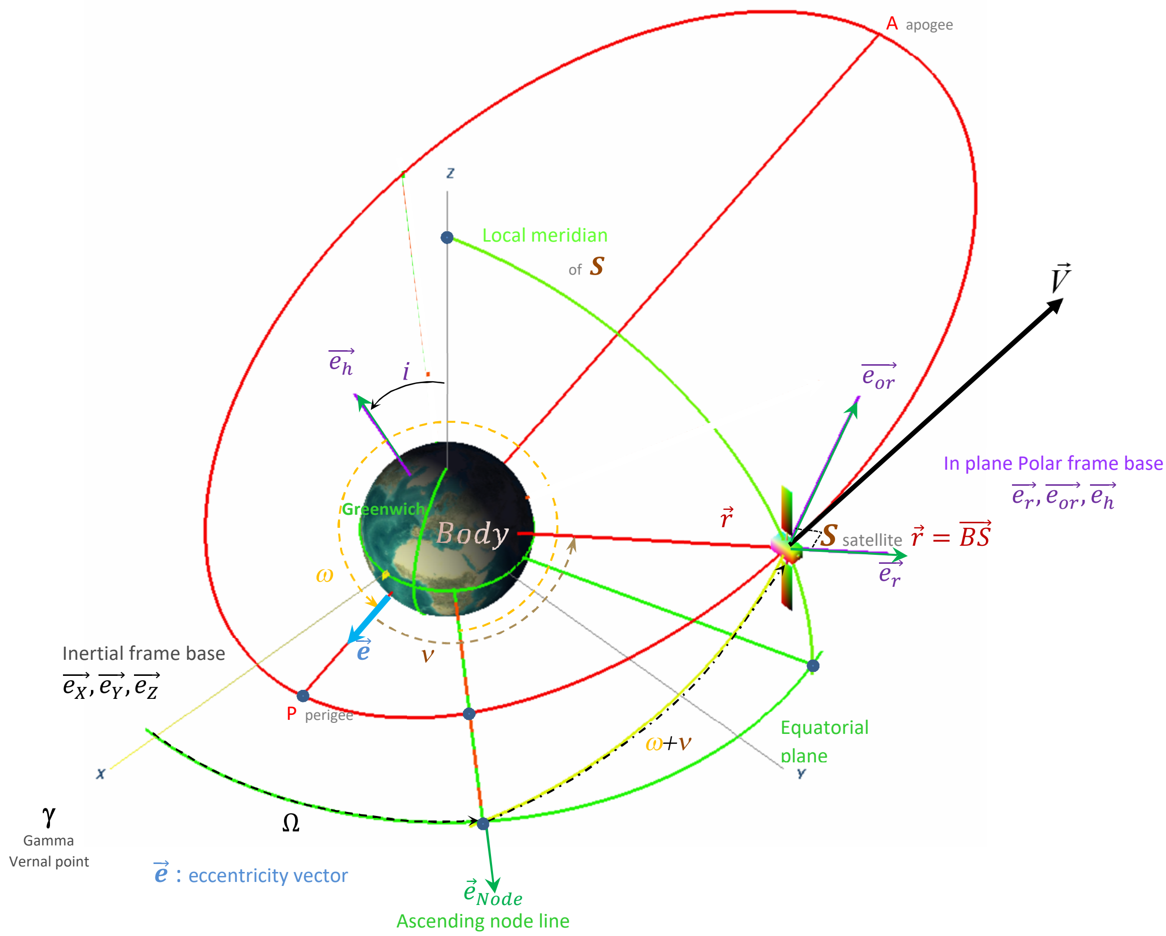
1 Orbital trajectory Frames and Angles

An elliptic (or hyperbolic) orbit in 3D space can be described with the help of 3 frames:

The body centered **inertial** frame X, Y, Z with base $\vec{e_X}, \vec{e_Y}, \vec{e_Z}$ and with X axis pointing in the (inertial) direction to the Sun the first day of spring, which is called the Vernal point $\boldsymbol{\gamma}$

And two useful frames referenced into the orbital plane, having the same out of plane base vector $\vec{e_h} = \frac{1}{\|\vec{H}\|}\vec{H}$ with $\vec{H} = \vec{r} \times \vec{V}$

- The **Polar** frame is : $\vec{r}, \vec{H} \times \vec{r}, \vec{H}$, with base $\vec{e_r}, \vec{e_{or}}, \vec{e_h}$ and $\vec{e_r} = \frac{1}{\|\vec{r}\|}\vec{r}$, $\vec{e_{or}}$ is the orthoradial base vector
- The **Nodal** frame based on the node $\vec{N} = \vec{e_z} \times \vec{H}$ with unit vector $\vec{e}_{Node} = \vec{N}/\|\vec{N}\|$.



2 Generalities Conics and Movement

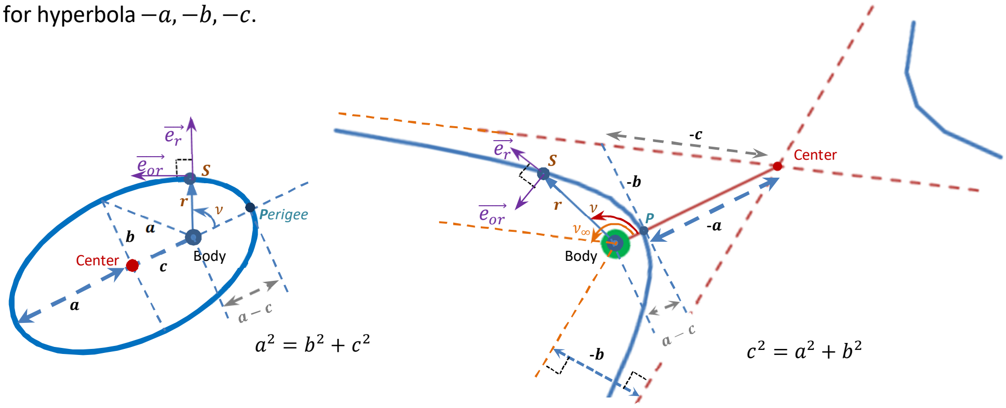
2.1 Recall some basic relations for conics

- Conic equation: the body at one focus (ν is the angle wrt the focal axis in the positive sense) : $r = \frac{p}{1+e\cos\nu}$ *Eq. 1*

  This is valid also for the hyperbola main branch (when $1 + e\cos\nu > 0$ i.e. with $\cos\nu_\infty = \frac{-1}{e}$ for $\nu \in]-\nu_\infty, \nu_\infty[$)

  For the other branch, the conic equation provides negative r values for angles ν such that $1 + e\cos\nu < 0$, i.e. $\nu \in]\nu_\infty, 2\pi - \nu_\infty[$.
- Semi-major axis: $a^2 = b^2 + c^2$ (for ellipses) and $c^2 = a^2 + b^2$ (for hyperbola) *Eq. 2*

- **The other equations for the ellipse can work as well for hyperbola on the condition to change the sign of the quantities semi-major axis, semi-minor axis and focus to center distance "a, b, c" .**
- For the case hyperbolic the distance between the Focus and the perigee "r_{per}" can be defined as for ellipses by $r_{per} = a - c$ with the fact that $a < 0$ and $c < 0$, so that r_{per} is a positive distance. This is shown in the figure below with the positive distances for hyperbola $-a, -b, -c$.



- For both cases, the eccentricity is : $e = {}^{c}/_{a}$ (for hyperbola, $a < 0$ and $c < 0$ so $e > 0$) *Eq. 3*
- For both cases, as given above: $r_{per} = a - c$ hence $r_{per} = a(1 - e)$ (for hyperbola, a<0, e>1 ➔ $r_{per} > 0$) *Eq. 4*
- For $\nu = 0$ $r(\nu = 0) = r_{per}$ the conic equation gives $r_{per} = \frac{p}{1+e}$ i.e. $p = r_{per}(1 + e)$ thus $p = a(1 - e^2)$ *Eq. 5*
- Eventually the conic equation becomes: $r = \frac{a(1-e^2)}{1+e\cos\boldsymbol{\nu}}$ *Eq. 6*

2.2 Recalls the general equation of the movement

The general equation of the movement, in a body centered inertial frame (the body is the Focus), under a central force (the gravitational force) position and velocity are related by $\vec{\Gamma} = {}^{\vec{F}}/_{m}$ with $\vec{F} = \frac{-\mu.m}{r^2}\vec{e_r}$ .In the "In-Plane" Polar frame $\vec{e_r}, \vec{e_{or}}, \vec{e_h}$: . *Eq. 7*

- $\vec{\boldsymbol{r}} = \begin{bmatrix} r \\ 0 \\ 0 \end{bmatrix}_{\vec{e_r},\vec{e_{or}},\vec{e_h}}$ $\quad \vec{\boldsymbol{V}} = \frac{d\vec{r}}{dt} = \frac{dr}{dt}\vec{e_r} + r\frac{d\vec{e_r}}{dt} = \begin{bmatrix} \frac{dr}{dt} \\ r\frac{d\theta}{dt} \\ 0 \end{bmatrix}$ $\quad \vec{\boldsymbol{\Gamma}} = \frac{d\vec{V}}{dt} = \begin{bmatrix} \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 \\ \frac{1}{r}\frac{d\left(r^2\frac{d\theta}{dt}\right)}{dt} \\ 0 \end{bmatrix} = {}^{\vec{\boldsymbol{F}}}/_{m} = \begin{bmatrix} \frac{-\mu}{r^2} \\ 0 \\ 0 \end{bmatrix}$ with θ a polar angle. *Eq. 8*
- It follows that the **angular momentum** per unit of mass $\vec{H} = \vec{r} \times \vec{V} = \begin{bmatrix} 0 \\ 0 \\ r^2\frac{d\theta}{dt} \end{bmatrix}$ has a null derivative because $\frac{1}{r}\frac{d\left(r^2\frac{d\theta}{dt}\right)}{dt} = 0$,

  then $\vec{H}$ is a constant vector aligned along $\vec{e_h}$, so $\vec{H} = r^2\frac{d\theta}{dt}\vec{e_h}$. One sets $H = \|\vec{H}\|$. And because $\frac{d\theta}{dt}$ is by definition *Eq. 9*

  always positive (if it looks negative it's because the inclination of the orbital plane is near 180°), $H = r^2\frac{d\theta}{dt}$, this gives the variable rate of evolution of the polar angle $\frac{d\theta}{dt} = \frac{H}{r^2}$. *Eq. 10*
- The power is $\vec{V}\cdot\vec{F}$, so the power per unit of mass is $\vec{V}\cdot\vec{\Gamma} = \vec{V}\cdot\frac{d\vec{V}}{dt} = \frac{d\left(\frac{V^2}{2}\right)}{dt} = \frac{-\mu}{r^2}\cdot\frac{dr}{dt} = \frac{d\left(\frac{\mu}{r}\right)}{dt}$ *Eq. 11*

  Thus, $\frac{d\left(\frac{V^2}{2} - \frac{\mu}{r}\right)}{dt} = 0$, this null derivative versus time shows that the sum of kinetic and gravitational energies is W a constant given by $\frac{V^2}{2} - \frac{\mu}{r} = W$ equation called "vis viva equation", see the complete eq. below . *Eq. 12*
- Because $\frac{d\vec{H}}{dt} = 0$, one consider $\frac{d\vec{V}\times\vec{H}}{dt}$ to get: $\frac{d\vec{V}\times\vec{H}}{dt} = \vec{\Gamma}\times\vec{H}$, with $\vec{\Gamma}\times\vec{H} = \begin{bmatrix} \frac{-\mu}{r^2} \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ r^2\frac{d\theta}{dt} \end{bmatrix} = \mu\frac{d\vec{e_r}}{dt}$ then $\frac{d(\vec{V}\times\vec{H} - \mu\vec{e_r})}{dt} = 0$

  The vector $\vec{e} = \frac{\vec{V}\times\vec{H}}{\mu} - \vec{e_r}$ derivative=0, so $\vec{e}$ is a constant vector. *Eq. 13*

  Note: $\vec{e}.\vec{H} = \frac{\vec{V}\times\vec{H}}{\mu}.\vec{H} + \vec{e_r}.\vec{H} = \frac{\vec{H}\times\vec{H}}{\mu}.\vec{V} + 0 = 0$, so vector $\vec{\boldsymbol{e}}$ is "in plane" the orbital plane, called **eccentricity vector**. *Eq. 14*

  Because $\vec{V}\times\vec{H}.\vec{r} = \vec{r}\times\vec{V}.\vec{H} = H^2$ $\quad \vec{e}.\vec{e_r} = \frac{\vec{V}\times\vec{H}}{\mu}.\vec{e_r} - 1$ $\quad \vec{e}.\vec{e_r} = \frac{H^2}{\mu\, r} - 1$ giving a general conic eq. $r = \frac{H^2/_\mu}{1+\vec{e}.\vec{e_r}}$. *Eq. 15*

  In this equation, all being constant except $\vec{e_r}$ and r, it is clear that <u>r is minimum</u> for $\vec{e_r}.\vec{e}$ maximum, i.e. $\vec{e_r} = \vec{e}/e$ <u>hence $\vec{e}$ is aligned with the perigee</u>. One can write $\vec{e}.\vec{e_r} = e\cos\boldsymbol{\nu}$ and $\vec{e}\times\vec{e_r}.\vec{e_h} = e\sin\boldsymbol{\nu}$. where $\boldsymbol{\nu}$ is the "in plane" angle from perigee (in the positive sense, on the focal axis) and by calling $p = \frac{H^2}{\mu}$ one gets the common conic equation: $r = \frac{p}{1+e\cos\nu}$ *Eq. 16*
- Energy constant: $r_{per} = \frac{p}{1+e} = \frac{H^2}{\mu(1+e)}$ and $\frac{{V_{per}}^2}{2} - \frac{\mu}{r_{per}} = W$ so $W = \frac{{r_{per}}^2{V_{per}}^2}{2{r_{per}}^2} - \frac{\mu}{r_{per}} = \frac{H^2}{2r_{per}\frac{H^2}{\mu(1+e)}} - \frac{\mu}{r_{per}} = \frac{\mu(1+e)}{2r_{per}} - \frac{\mu}{r_{per}}$

  $W = -\frac{\mu}{2r_{per}}\big(2 - (1+e)\big) = -\frac{\mu}{2\frac{p}{1+e}}(1 - e) = -\frac{\mu}{2p}(1 - e^2)$ with $p = a(1 - e^2)$ $W = \frac{-\mu}{2a}$ *Eq. 17*

  The completed "**vis viva equation**" becomes: $\frac{V^2}{2} - \frac{\mu}{r} = \frac{-\mu}{2a}$ *Eq. 18*

3 Transformation $\vec{r}, \vec{V}$ to Orbital elements $a, e, i, \Omega, \omega, \nu$

One considers the vectors $\vec{r}, \vec{V}$ given in the inertial base $\overrightarrow{e_X}, \overrightarrow{e_Y}, \overrightarrow{e_Z}$ of the body centered inertial frame, $\overrightarrow{e_X}$ pointing to γ.

$\overrightarrow{e_r} = \frac{1}{\|\vec{r}\|}\vec{r} \qquad r = \|\vec{r}\| \qquad V = \|\vec{V}\|$ Eq. 19

The vis viva equation $\frac{V^2}{2} - \frac{\mu}{r} = -\frac{\mu}{2a}$ gives $a = \frac{-\mu}{2\left(\frac{V^2}{2} - \frac{\mu}{r}\right)}$ (for hyperbola, $a < 0$) Eq. 20

$\vec{H} = \vec{r} \times \vec{V}, \quad \overrightarrow{e_h} = \frac{1}{\|\vec{H}\|}\vec{H} \qquad \cos(i) = \overrightarrow{e_h}.\overrightarrow{e_Z}$ that gives inclination i without any ambiguities because $i \in [0°, 180°]$. Eq. 21

Using the eccentricity vector $\vec{e} = \frac{\vec{V} \times \vec{H}}{\mu} - \overrightarrow{e_r} \qquad e = \|\vec{e}\|$ which is always ≥ 0 Eq. 22

- If $i \in]0°, 180°[$ then the node $\vec{N} = \overrightarrow{e_z} \times \vec{H}, \quad \vec{e}_{Node} = \frac{1}{\|\vec{N}\|}\vec{N}$. Angle Ω is the angle of the node $\vec{N}$ wrt $\overrightarrow{e_X}$

  $\cos(\Omega) = \vec{e}_{Node}.\overrightarrow{e_X} \qquad \sin(\Omega) = \vec{e}_{Node}.\overrightarrow{e_Y}$ that gives $\Omega = ATAN2(\cos, \sin)$ Eq. 23
  - If $e > 0$, the angle ω is the angle of $\vec{e}$ wrt the node : $\vec{e}.\vec{e}_{Node} = e\cos\omega$; $\vec{e}_{Node} \times \vec{e}.\vec{e}_h = e\sin\omega$ gives ω Eq. 24
  - If $e > 0$, using again $\vec{e}$, the angle ν is the angle of $\overrightarrow{e_r}$ wrt $\vec{e}$: $\vec{e}.\overrightarrow{e_r} = e\cos\nu$; $\vec{e} \times \overrightarrow{e_r}.\vec{e}_h = e\sin\nu$ gives ν Eq. 25

    Note : $\vec{e}_{Node} \times \vec{e}.\vec{e}_h = \vec{e} \times \vec{e}_h.\vec{e}_{Node} = -\vec{e}_h \times \vec{e}.\vec{e}_{Node}$ and $\vec{e} \times \overrightarrow{e_r}.\vec{e}_h = \vec{e}_h \times \vec{e}.\overrightarrow{e_r}$ so $\vec{e}_h \times \vec{e}$ oriented along the semi-minor axis of the conic can be used for both assessments of $\sin\omega$ and $\sin\nu$.
  - If $e = 0$, then only $\omega + \nu$ is known, ω can be set to zero.
- Else, if $i = 0$ or $i = 180°$ then the node and Ω are undefined, the orbit is in the plane $\overrightarrow{e_X}, \overrightarrow{e_Y}$ (i.e. equatorial) , the angle wrt the axis X given by $cos = \vec{r}.\overrightarrow{e_X} \quad sin = \vec{r}.\overrightarrow{e_Y}$ represents $\Omega + \omega + \nu = ATAN2(\cos, \sin)$.
  - if $e > 0$ Ω can be set to zero
  - if $e = 0$ then the perigee and ω are undefined, Ω and ω can be set to zero.

4 Transformation Orbital elements $a, e, i, \Omega, \omega, \nu$ to $\vec{r}, \vec{V}$ in body centred inertial frame

Note : Case elliptic: from r_{per}, r_{apo} one gets $a = \frac{(r_{per} + r_{apo})}{2}$ Hyperbolic: from V_∞ one gets a with $\frac{V_\infty{}^2}{2} = \frac{-\mu}{2a} \quad a < 0$

1) Radius vector: one sets $\vec{r} = r.\overrightarrow{e_r}$ and $r = \|\vec{r}\|$ and "$p = a(1 - e^2)$" Eq. 26

$$r = \frac{p}{1 + e\cos\nu} \qquad \overrightarrow{e_r} = \begin{bmatrix} \cos\Omega\cos(\omega+\nu) + \cos\left(\Omega + \frac{\pi}{2}\right)\sin(\omega+\nu)\cos i \\ \sin\Omega\cos(\omega+\nu) + \sin\left(\Omega + \frac{\pi}{2}\right)\sin(\omega+\nu)\cos i \\ \sin(\omega+\nu)\sin i \end{bmatrix}$$ Eq. 27

2) Velocity vector

$$\overrightarrow{V(t)} = \frac{d\vec{r}}{dt} = \frac{d\vec{r}}{d\nu}.\frac{d\nu}{dt} = \left(\frac{dr}{d\nu}\overrightarrow{e_r} + r\frac{d\overrightarrow{e_r}}{d\nu}\right).\frac{d\nu}{dt}$$ Eq. 28

with $H = \sqrt{\mu a(1 - e^2)}$ from "$p = a(1 - e^2) = \frac{H^2}{\mu}$" and $\frac{d\nu}{dt} = \frac{H}{r^2}$ from "$H = r^2\frac{d\theta}{dt}$", $\overrightarrow{V(t)} = \left(e\sin\nu\frac{r^2}{p}\overrightarrow{e_r} + r\frac{d\overrightarrow{e_r}}{d\nu}\right)\frac{H}{r^2}$ Eq. 29

$\overrightarrow{V(t)} = \frac{\mu\, e\sin\nu}{H}\overrightarrow{e_r} + \frac{H}{r}\frac{d\overrightarrow{e_r}}{d\nu}$ giving eventually $\overrightarrow{V(t)} = \frac{H}{r}\frac{e\sin\nu}{a(1-e^2)}\vec{r} + \frac{H}{r}\frac{d\overrightarrow{e_r}}{d\nu}$ Eq. 30

with

$$\frac{d\overrightarrow{e_r}}{d\nu} = \begin{bmatrix} -\cos\Omega\sin(\omega+\nu) + \cos\left(\Omega + \frac{\pi}{2}\right)\cos(\omega+\nu)\cos i \\ -\sin\Omega\sin(\omega+\nu) + \sin\left(\Omega + \frac{\pi}{2}\right)\cos(\omega+\nu)\cos i \\ \cos(\omega+\nu)\sin i \end{bmatrix}$$ Eq. 31

5 References:

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La mécanique orbitale est une discipline étrange… La première fois que vous la découvrez, vous ne comprenez rien.. La deuxième fois, vous pensez que vous comprenez, sauf un ou deux points.. La troisième fois, vous savez que vous ne comprenez plus rien, mais à ce niveau vous êtes tellement habitué que ça ne vous dérange plus. attribué à Arnold Sommerfeld pour la thermodynamique, vers 1940

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